Lecture 5

Properties of groups

T1. Each a element of group G has one and only one symmetrical (inverse) element $a' \in G$: $a \circ a' = e$.

Proof. Let's assume that a has two different inverse elements a_1', a_2':

$$a \circ a_1' = e \quad , \quad a \circ a_2' = e$$

Then

$$a_2' \circ (a \circ a_1') = a_2' \circ e = a_2'$$
$$(a_2' \circ a) \circ a_1' = e \circ a_1' = a_1'$$
$$\Rightarrow a_1' = a_2'$$

$\triangleright$

T2. For each $a \in G$:

$$(a')' = a.$$

Proof. For inverse elements we have

$$a' \circ a = a \circ a' = e$$

Thus $(a')' = a$ since ~~so~~ for any element $b \in G$ the inverse is unique. ▷

T3. $(a_1 \circ a_2)' = a_2' \circ a_1'$

Proof: We test if $a_2' \circ a_1'$ is the inverse element of $a_1 \circ a_2$

$$a_1 \circ a_2 \circ (a_2' \circ a_1') = a_1 \circ (a_2 \circ a_2') \circ a_1'$$

$$= a_1 \circ e \circ a_1' = a_1 \circ a_1' = e.$$ ▷

Now we define two more operations

1. $a - b := a + b' = a + (-b)$

2. $a / b := a \cdot b' = a \cdot b^{-1}$.

($(-b)$ and b^{-1} are inverse elements for $+$ and $\cdot$ oper.)

T. If group G is abelian then (similar results are valid for $+$ operation).

1. $(a/b) \cdot (c/d) = (a \cdot c)/(b \cdot d)$

2. $(a/b)' = b/a$

3. $(a/b) \cdot (c/d)' = (a \cdot d)/(b \cdot c)$

Proof.

2. $(a/b)' \overset{def.}{=} (a\cdot b')'$

$\overset{T3}{=} (b')'\cdot a' \overset{T2}{=} b\cdot a' \overset{def}{=} b/a$

1. $(a/b)\cdot(c/d) \overset{def.}{=} (a\cdot b')\cdot(c\cdot d')$

$\overset{abel}{=} a\cdot(c\cdot b')\cdot d' = (a\cdot c)\cdot(b'\cdot d')$

$\overset{abel}{=} (a\cdot c)\cdot(d'\cdot b') \overset{T3}{=} (a\cdot c)\cdot(b\cdot d)'$

$= (a\cdot c)/(b\cdot d)$ ▷

In many cases we have different but similar (similar structure) groups. It is important to define a level of this similarity

Group homomorphism

__Def.__ Given ~~two~~ groups

$(G, *)$ and $(H, \circ)$,

a group homomorphism from $(G, *)$ to $(H, \circ)$ is a function

$$h : G \to H$$

such that for all $u, v \in G$ it holds that

$$h(u * v) = h(u) \circ h(v)$$

($\forall u, v$ we have that $h(u) \in H$, $h(v) \in H$).

__Properties:__ Give a proof that

1) $h(e_G) = e_H$ (identity elements)

2) $h(u^{-1}) = h(u)^{-1}$ (inverses)

Group isomorphism

<u>Def.</u> Given ~~two~~ groups $(G, *)$ and $(H, \circ)$, a group isomorphism from $(G, *)$ to $(H, \circ)$ is a bijective group homomorphism from G to H, i.e. there exists a bijective function $f: G \to H$ such that

$$f(u * v) = f(u) \circ f(v), \quad u, v \in G.$$

Notation: $G \cong H$ (or $(G, *) \cong (H, \circ)$).

<u>Example</u> $G_1 = (1\mathbb{Z}, +)$ $\quad * = +$

$G_2 = (3\mathbb{Z}, +)$, $\quad \circ = +$

Operations of both groups are the same.

Let's show that groups are homomorphic.

$h: a \to 3a \quad \forall a \in \mathbb{Z}$.

$\forall a \in \mathbb{Z} \implies 3a \in 3\mathbb{Z}$.

$$h(a \overset{+_1}{+} b) = 3(a+b) = 3a \overset{+_2}{+} 3b$$

$$\begin{array}{r} ||| \\ h(a) \overset{+_2}{+} h(b) \end{array}$$

Since h is also <u>bijection</u>, then we get that both groups are <u>isomorphic</u>.

$3a \xrightarrow{h^{-1}} a$

Let's consider two groups

$$G_1 = (\mathbb{R}, +), \quad G_2 = (\mathbb{R}, \cdot)$$

They are ~~isomofosfic~~ isomorphic.

This isomorphism is defined by function $\boxed{f(x) = \ln x}$

$$\ln(a \cdot b) = \ln a + \ln b,$$
$$a, b \in \mathbb{R}_+.$$